LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **APRIL 2016**

MT 1815 - LINEAR ALGEBRA

Date: 28-04-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer ALL the questions.

I. a) i) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent.
1. c is a characteristic value of T.
2. The operator $(T-cI)$ is singular.
3. det $(T-cI)=0$.

(OR) (5)

ii) Prove that similar matrices have the same characteristic polynomials.

b) i) Let T be a linear operator on a finite dimensional vector space V. Prove that the minimal polynomial for T divides the characteristic polynomial for T.

(OR)

ii) Let V be a finite dimensional vector space over F and T be a linear operator on V. Then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.
(15)

II. a. i) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then show that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \dots (x - c_k)$, where $c_1, \dots, c_k$ are distinct elements of F.

(OR) (5)

ii) Let W be an invariant subspace for T. Then prove that the minimal polynomial for T_w divides the minimal polynomial for T.

b) i) State and prove Primary Decomposition theorem.

(OR) (15)

ii) Let T be linear operator on a finite dimensional space V and $c_1, \dots c_k$ be the distinct characteristic values of T. Prove that T is diagonalizable if and only if there exist k linear operators $E_1, \dots E_k$ on V such that
1. Each E_i is a projection.
2. $E_i E_j = 0, i \neq j$.
3. $I = E_1 + \dots + E_k$.
4. $T = c_1 E_1 + \dots + c_k E_k$
5. The range of E_i is the characteristic space of T associated with c_i.

III. a) i) Let T be a linear operator on a vector space V and W a proper T-admissible subspace of V. Prove that $W \cap Z(\alpha;T) = \{0\}$.

(OR) (5)

ii) If U is a linear operator on a finite dimensional space W, then prove that U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of the minimal polynomial for U.

b) (i) Stat and Prove cyclic decomposition theorem.

(OR) (15)

(ii)) Let α be any non-zero vector in V and let p_α be the T-annihilator of α . Prove the following statements:

1. The degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha;T)$.
2. If the degree of p_α is k, then the vectorsα, $T\alpha$, $T^2\alpha$,…$T^{k-1}\alpha$ form the basis for $Z(\alpha;T)$.
3. If U is the linear operator on $Z(\alpha;T)$ induced by T, then the minimal polynomial for U is p_α .

IV. a) i) Define the matrix of a form on a real or complex vector space with respect to any ordered basis . Let f be the form on R^2 defined by $f((x_1,y_1),(x_2,y_2)) = x_1y_1 + x_2y_2$. Find the matrix of f with respect to a basis {(1,-1), (1, 1)}.

(OR) (5)

ii) Let T be a linear operator on a complex finite dimensional inner product space V. Then prove that T is self-adjoint if and only if $\langle T\alpha,\alpha\rangle$ is real for every α in V.

b) i) For any linear operator T on a finite-dimensional inner product space V, show that there exists a unique linear T* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all α, β in V. (7)

ii) Prove that for every Hermitian form f on a finite-dimensional inner product space V, there is an orthonormal basis of V in which f is represented by a diagonal matrix with real entries. (8)

(OR)

iii) Let F be the field of real numbers or the field of complex numbers. Let A be an n x n matrix over F. Show that the function g defined by g (X, Y) = Y* AX is a positive form on the space F^{nx1} if an only if there exists and invertible n x n matrix P with entries in F such that A = P*P.

iv) State and prove Principal Axis Theorem. (7+8)

V. a. i) Let V be a complex vector space and f be a bilinear form on V such that $f(\alpha, \alpha)$ is real for every α. Then prove that f is Hermitian.

(OR) (5)

ii) Define the quadratic form q associated with a symmetric bilinear form f and prove that $f(\alpha,\beta) = \frac{1}{4}q(\alpha+\beta) - \frac{1}{4}q(\alpha-\beta)$.

b) i) Let V be a n-dimensional vector space over the field of real numbers and let f be a symmetric bilinear form on V which has rank r. Then show that there is an ordered basis B= $\{\beta_1,\ldots,\beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \beta_j) = \pm 1$, j = 1, …,r. Furthermore, the number of basis vectors β_j for which $f(\beta_j, \beta_j) = 1$ is independent of the choice of basis.

(OR) (15)

ii) If f is a non-zero skew-symmetric bilinear form on a finite dimensional vector space V, then prove that there exist a finite sequence of pairs of vectors, $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ...(\alpha_k, \beta_k)$ with the following properties:

1) f (α_j, β_j)=1, j=1,2,,…,k.

2) f (α_i, α_j)=f (β_i, β_j)=f (α_i, β_j)=0,i $\neq$ j.

3) If W_j is the two dimensional subspace spanned by α_j and β_j, then V=$W_1 \oplus W_2 \oplus ...W_k \oplus W_0$ where W_0 is orthogonal to all α_j and β_j and the restriction of f to W_0 is the zero form.

(15)