LOYOLA COLLEGE (AUTONOMOUS), CHENNAI - 600 034

M.Sc. DEGREE EXAMINATION - MATHEMATICS

FIRST SEMESTER - APRIL 2016

MT 1815 - LINEAR ALGEBRA

Date: 28-04-2016 Time: 01:00-04:00	Dept. No.	Max.: 100 Marks
Time: 01.00 01.00		

Answer ALL the questions.

- I. a) i) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent.
 - 1. c is a characteristic value of T.
 - 2. The operator (*T-cI*) is singular.
 - 3. $\det(T-cI) = 0$.

$$(OR) (5)$$

- ii) Prove that similar matrices have the same characteristic polynomials.
- b) i) Let T be a linear operator on a finite dimensional vector space V. Prove that the minimal polynomial for T divides the characteristic polynomial for T.

(OR)

- ii) Let V be a finite dimensional vector space over F and T be a linear operator on V. Then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

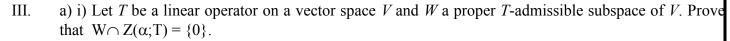
 (15)
- II. a. i) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then show that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x c_1) \dots (x c_k)$, where c_1, \dots, c_k are distinct elements of F.

$$(OR) (5)$$

- ii) Let W be an invariant subspace for T. Then prove that the minimal polynomial for T_w divides the minimal polynomial for T.
- b) i) State and prove Primary Decomposition theorem.

$$(OR) (15)$$

- ii) Let T be linear operator on a finite dimensional space V and $c_1, \ldots c_k$ be the distinct characteristic values of T. Prove that T is diagonalizable if and only if there exist k linear operators $E_1, \ldots E_k$ on V such that
 - 1. Each E_i is a projection.
 - 2. $E_i E_j = 0, i \neq j$.
 - 3. $I = E_1 + ... + E_k$.
 - 4. $T = c_1 E_1 + ... + c_k E_k$
 - 5. The range of E_i is the characteristic space of T associated with c_i



$$(OR) (5)$$

- ii) If U is a linear operator on a finite dimensional space W, then prove that U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of the minimal polynomial for U.
- b) (i) Stat and Prove cyclic decomposition theorem.

$$(OR) (15)$$

- (ii) Let α be any non-zero vector in V and let P_{α} be the *T*-annihilator of α . Prove the following statements:
 - 1. The degree of P_{α} is equal to the dimension of the cyclic subspace $Z(\alpha;T)$.
 - 2. If the degree of P_{α} is k, then the vectors α , $T\alpha$, $T^{2\alpha}$,... $T^{k-1}\alpha$ form the basis for $Z(\alpha;T)$.
 - 3. If U is the linear operator on $Z(\alpha;T)$ induced by T, then the minimal polynomial for U is P_{α} .
- IV. a) i) Define the matrix of a form on a real or complex vector space with respect to any ordered basis. Let f be the form on R^2 defined by $f((x_1, y_1), (x_2, y_2)) = x_1y_1 + x_2y_2$. Find the matrix of f with respect to a basis $\{(1,-1), (1,1)\}$.

$$(OR) (5)$$

- ii) Let T be a linear operator on a complex finite dimensional inner product space V. Then prove that T is self-adjoint if and only if $\langle T\alpha, \alpha \rangle$ is real for every α in V.
- b) i) For any linear operator T on a finite-dimensional inner product space V, show that there exists a unique linear T^* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all α , β in V. (7)
 - ii) Prove that for every Hermitian form f on a finite-dimensional inner product space V, there is an orthonormal basis of V in which f is represented by a diagonal matrix with real entries.

(OR)

iii) Let F be the field of real numbers or the field of complex numbers. Let A be an n x n matrix over F. Show that the function g defined by $g(X, Y) = Y^* AX$ is a positive form on the space F^{nx1} if an only if there exists and invertible n x n matrix P with entries in F such that $A = P^*P$.

iv) State and prove Principal Axis Theorem. (7+8)

V. a. i) Let V be a complex vector space and f be a bilinear form on V such that $f(\alpha, \alpha)$ is real for every α . Then prove that f is Hermitian.

$$(OR) (5)$$

(8)

- ii) Define the quadratic form q associated with a symmetric bilinear form f and prove that $f(\alpha, \beta) = \frac{1}{4}q(\alpha + \beta) \frac{1}{4}q(\alpha \beta)$.
- b) i) Let V be a n-dimensional vector space over the field of real numbers and let f be a symmetric bilinear form on V which has rank r. Then show that there is an ordered basis $B = \{\beta_1,...,\beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \beta_j) = \pm 1$, j = 1, ..., r. Furthermore, the number of basis vectors β_i for which $f(\beta_i, \beta_i) = 1$ is independent of the choice of basis.

 $(OR) \tag{15}$

ii) If f is a non-zero skew-symmetric bilinear form on a finite dimensional vector space V, then prove that there exist a finite sequence of pairs of vectors, $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ...(\alpha_k, \beta_k)$ with the following properties:

- 1) $f(\alpha_i, \beta_i)=1, j=1,2,...,k$.
- 2) $f(\alpha_i, \alpha_j) = f(\beta_i, \beta_j) = f(\alpha_i, \beta_j) = 0, i \neq j.$
- 3) If W_j is the two dimensional subspace spanned by α_j and β_j , then $V=W_1\oplus W_2\oplus...W_k\oplus W_0$ where W_0 is orthogonal to all α_j and β_j and the restriction of f to W_0 is the zero form.

(15)