LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2016**

MT 2810 - ALGEBRA

Date: 20-04-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

ANSWER ALL QUESTIONS.

I a) Show that any group of order 45 has a subgroup of order 9.

[OR]

b) Let G be a group and $a \in G$. Prove that $N(a)$ is a subgroup of G. **(5)**

c i) State and prove Cauchy's theorem.

ii) If $o(G) = p^n$ where p is a prime number then prove that $Z(G) \neq (e)$.

(8+7)

[OR]

d i) State and prove Sylow's first theorem.

ii) Show that $Z_{12} \approx Z_3 \oplus Z_4$ **(10+5)**

II a) Show that $x^6 + 9x^5 - 12x^4 + 24x^3 - 15x + 12$ is irreducible over rational numbers.

[OR]

b) If $f(x)$ *and* $g(x)$ are two polynomials in $F[x]$ then prove that $deg f(x)g(x) = deg f(x) + deg g(x)$. **(5)**

c i) State and prove the Eisentein criterion.

ii) State and prove division algorithm **(10+5)**

[OR]

d i) State and prove Gauss lemma.

ii) Prove that the polynomial $x^2 + 1$ is irreducible over the field F of integers mod 11and show that $F[x]/x^2 + 1$ is a a field having 121 elements. **(7+8)**

III a) Find the degree of $\sqrt{2} + \sqrt{3}$ over Q .

[OR]

b) State and prove remainder theorem. **(5)**

c) If L is a finite extension of K and K is a finite extension of F then prove that L is a finite extension of F and $[L:F] = [L:K][K:F]$.

[OR]

d) Let $f(x) \in F[x]$ be of degree $n \geq 1$. Prove that there exist an extension E of F of degree at most $n!$ in which $f(x)$ has n roots. **(15)**

IV a) Let F be the field of rational numbers and let $f(x) = x^4 + x^2 + 1$ □ $F[x]$ .Find the splitting field of $f(x)$.

[OR]

b) Define a normal extension and give an example. **(5)**

c) Let F be a field and let $F(x_1, \ldots\ldots x_n)$ be the field of rational functions in $x_1, \ldots\ldots x_n$ over F. Suppose S is the Field of symmetric rational functions. Then prove that

(i) $[F(x_1, \ldots\ldots x_n) : S] = n!$

(ii) $G([F(x_1, \ldots\ldots x_n) : S]) = S_n$

(iii) If $a_1, \ldots\ldots a_n$ are elementary symmetric functions in $x_1, \ldots\ldots x_n$ then $S = F(a_1, \ldots\ldots a_n)$.

(iv) $F(x_1, \ldots\ldots x_n)$ is the splitting field over $F(a_1, \ldots\ldots a_n)$ =S of the polynomial $t^n - a_1 t^{n-1} + a_2 t^{n-2} \ldots + (-1)^n a_n$

[OR]

d) State and prove Fundamental theorem of Galois theory . **(15)**

V a) Let F be a finite field with q elements and $F \subset K$ where K is also a finite field. Prove that K has q^n elements where $n = [K:F]$.

[OR]

b) Derive the cyclotomic polynomials $\Phi_3(x)$ and $\Phi_4(x)$. **(5)**

c) Let G be a finite abelian group satisfying the relation $x^n = e$ is satisfied by at most n elements of G, for every integer n. Then prove that G is a cyclic group.

[OR]

d) State and prove Wedderburn's theorem on division rings. **(15)**

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