LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2016**

MT 2811 - MEASURE THEORY AND INTEGRATION

Date: 22-04-2016 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer **ALL** questions:

1. (a) For any sequence of sets $\{E_i\}$, prove that $m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.

(OR)

(b) Prove that the class M of Lebesgue measurable sets is a σ-algebra. **(5)**

(c) (i) Prove that every interval is measurable.

(ii) Prove that there exists a non-measurable set. **(5+10)**

(OR)

(d) Prove that the following statements regarding the set E are equivalent:

(i) E is measurable.

(ii) $\forall \epsilon > 0$, there exists an open set $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$.

(iii) $\exists G$, a G_δ −set, $G \supseteq E$ such that $m^*(G - E) = 0$.

(iv) $\forall \epsilon > 0$, $\exists F$, a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \epsilon$.

(v) $\exists F$, a F_σ −set, $F \subseteq E$ such that $m^*(E - F) = 0$. **(15)**

2. (a) Show that $\int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x} dx = 9\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$.

(OR)

(b) Let A and B be any two disjoint measurable sets. If φ is a simple function then prove that

(i) $\int_{A\cup B} \phi dx = \int_A \phi dx + \int_B \phi dx$

(ii) $\int a\phi dx = a\int \phi dx$, if a > 0. **(5)**

(c) State and prove Lebesgue's Monotone Convergence Theorem. **(15)**

(OR)

(d) (i) Prove that the following statements are equivalent:

1) f is a measurable function,

2) $\forall\alpha$, $[x: f(x) \geq \alpha]$ is measurable,

3) $\forall\alpha$, $[x: f(x) < \alpha]$ is measurable,

4) $\forall\alpha$, $[x: f(x) \leq \alpha]$ is measurable.

(ii) If f is Riemann integrable and bounded over the finite interval $[a, b]$, then prove that f is integrable $R\int_a^b f dx = \int_a^b f dx$. **(8+7)**

3. (a) Show that every algebra is a ring and every σ-algebra is a σ-ring.

(OR)

(b) If c is a real number and f, g are measurable functions, then prove that $f + g$ and fg are also measurable. **(5)**

(c) Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let S^* denote the class of μ^* -measurable sets. Then prove that S^* is a σ −ring and μ^* restricted to S^* is a Complete Measure.

(OR)

(d) Prove that the outer measure μ^* on $\mathcal{H}(\square)$ defined by $\mu^*(E) = \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R},\right.$

$\left. n = 1,2,\ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$ and the corresponding outer measure defined by $\bar{\mu}$ on $S(\mathfrak{R})$ and $\bar{\mu}$ on

S^* are the same. **(15)**

4. (a) Let ψ be strictly convex, then prove that ψ(∫ fdμ) = ∫ ψ • f $d\mu$ if and only if f = ∫ f dμ a.e.

(OR)

(b) Define a convex function and prove that for a convex function ψ on (a, b) such that a<s<t<u<b, then ψ(s, t) ≤ ψ(s, u). **(5)**

(c) (i) Let ψ be a function on (a, b). Then prove that ψ is convex on (a, b) if and only if for each *x* and *y* such that $a < x < y < b$, the graph of ψ on (a, x) and (y, b) does not lie below the line passing through $(x, \psi(x))$ and $(y, \psi(y))$.

(ii) Let $\{f_n\}$ be a sequence of non-negative measurable functions and let f be a measurable function such that $f_n \to f$ in measure, then prove that $\int f d\mu \le \lim inf \int f_n d\mu$. **(7+8)**

(OR)

(d) (i) State and prove Holder's Inequality.

(ii) State and prove Jensen's Inequality. **(8+7)**

5. (a) Let *v* be a signed measure and let μ, λ be measure on [X, S] such that μ, λ, v are σ - finite, $v \ll \mu$, $\mu \ll \lambda$ then prove that $\frac{dv}{d\lambda} = \frac{dv}{d\mu}\frac{d\mu}{d\lambda}[\lambda]$.

(OR)

(b) Prove that the countable union of sets with respect to a signed measure v is a positive set. **(5)**

(c) (i) Let *v* be a signed measure on $[\![X, S]\!]$. Then prove that there exists a positive set *A* and a negative set *B* such that $A \cup B = X$, $A \cap B = \Phi$. Prove further that it is unique to the extent that if A_1, B_1 and A_2, B_2 are Hahn decomposition of *X* with respect to *v*, then $A_1 \Delta A_2$ is a *v*-null set.

(ii) Define Signed measure and total variation of a signed measure. **(11+4)**

(OR)

(d) State and prove Radon-Nikodym Theorem. **(15)**

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