LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRSTSEMESTER – APRIL 2017

MT 1816- REAL ANALYSIS

Date: 02-05-2017 Dept. No. Max. : 100 Marks
09:00-12:00

Answer all Questions. All questions carry equal marks.

1. (a) State and prove the theorem on integration by parts and use the theorem to prove the following: Suppose f is a real continuously differentiable function on [a, b], f (a) =f (b) =0 and $\int_a^b f^2(x)dx = 1$. Prove that $abxfxf'xdx=-12$.

(OR)

(b) Prove that $\int_{a\_}^{b} f d\alpha \leq \int_a^{b-} f d\alpha$. (5 marks)

(c) i) State and prove the necessary and sufficient condition for a function f to be Riemann-Steiltjesintegrable with respect to α. (10 marks)

ii) Any monotone function f: [0, 1]→R is Riemann Integrable. Justify. (5 marks)

(OR)

(d) i) Suppose f∈ R(∝) on [a, b], $m \leq f \leq M$, ϕ is continuous on [m, M] and h(x) =ϕ(f(x)) on [a, b]. Then prove that h∈ R(∝) on [a, b]. (10 marks)

ii) If f∈ R(∝)and g∈ R(∝) on [a, b] then prove that

1) fg∈ R(∝)

2) $|f| \in$ R(∝)and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$. (5 marks)

2. (a) Prove that for $f_n(x) = n^2 x(1-x^2)^n, 0 \leq x \leq 1, n = 1,2 ..,$

$$\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx \neq \lim_{n\to\infty} \int_0^1 f_n(x) dx$$

(OR)

(b) State and prove the Cauchy criterion for uniform convergence of sequence of functions. (5 marks)

(c) State and prove the Stone-Weierstrass theorem.

(OR)

(d) If $\{f_n\}$ is a sequence of continuous functions on a set E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E. (15 marks)

3. (a) Let $\{\emptyset_0, \emptyset_1 \dots\}$ be orthonormal on I and assume that f$\in L^2(I)$. If the sequence of functions $\{s_n\}$ and $\{t_n\}$ on I are defined by $s_n(x) = \sum_{k=0}^{n} c_k \emptyset_k(x), t_n(x) = \sum_{k=0}^{n} b_k \emptyset_k(x)$, where $c_k = (f, \emptyset_k)$, k=0,1. . n and $b_0, b_1 \dots b_n$ are arbitrary complex numbers then for each n, prove that $||f - s_n|| \leq ||f - t_n||$ and the equality holds if and only if $b_k = c_k \; for \; k = 0,1 \dots n$.

(OR)

(b) State and prove the Bessel's Inequality and Parseval's formula. (5 marks)

(c) State and prove the Riemann-Lebesgue lemma and use the lemma to prove the following:

For f$\in L(-\infty, +\infty)$, $\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} f(t) \frac{1-\cos\alpha t}{t} dt = \int_0^{\infty} \frac{f(t)-f(-t)}{t} dt$. (15 marks)

(OR)

(d) (i) Define Dirichlet's kernel and prove that $\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin((2n+1)\frac{x}{2})}{2\sin\frac{x}{2}}, x \neq 2m\pi$

(ii) If $f \in L[0,2\pi]$, f is periodic with period 2π and $\{s_n\}$ is a sequence of partial sums of Fourier series generated by f, $s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), n = 1,2 \dots$ then prove that $s_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t) dt$ (5+10 marks)

4. (a) Prove that the set of all linear transformations of R^n into R^m, $L(R^n, R^m)$ is a metric space with respect to the norm defined by $||A|| = sup_{|x|\leq 1} |Ax|$, where $A \in L(R^n, R^m)$.

(OR)

(b) If Ω is the set of all invertible linear operators on R^n and for $A \in \Omega, B \in L(R^n)$, if$||B - A||||A^{-1}|| < 1$, then prove that $B \in \Omega$. (5 marks)

(c) State and prove the inverse function theorem.

(OR)

(d) State and prove the implicit function theorem. (15 marks)

5. (a) A cup of tea at $98^0 C$ is placed inside a room with its room temperature $72^0 C$, after 3 minutes its temperature reduces to $90^0 C$, at what time the temperature will reduce to $80^0 C$.

(OR)

(b) Explain rectilinear coordinate system with algebraic and geometric approach. (5 marks)

(c) Derive the expression for Newton's Law of Cooling.

(OR)

(d) Derive the D' Alembert's wave equation for a vibrating string. (15 marks)