LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc.DEGREE EXAMINATION –**MATHEMATICS**

FOURTH SEMESTER – **APRIL 2018**

MT 4810– FUNCTIONAL ANALYSIS

Date: 18-04-2018 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer ALL the questions:

1. (a) Prove that the union of two subspaces of a vector space X is a subspace of X if and only if one is contained in the other. **(5)**

(OR)

(b) If X is a vector space, Y and Z are subspaces of X and Y is complementary to Z, then prove that every element of X/Y contains exactly one element of Z. **(5)**

(c) Let X be a real vector space, let Y be a subspace of X and p be a real valued function on X such that $p(x+y) \leq p(x) + p(y)$ and $p(ax) = ap(x)$ for all $x, y \in X$, for $a \geq 0$. If f is a linear functional on Y and $f(x) \leq p(x)$ for all $x \in Y$, prove that there is a linear functional F on X such that $F(x) = f(x)$ for all $x \in Y$and $F(x) \leq p(x)$ for all $x \in X$.

(15)

(OR)

(d) (i) Prove that there is a natural isomorphism between a subspace of $X^{**}$and X itself.

(ii) Prove that if $f \in X^*$, then $Z(f)$ has deficiency 0 or 1 in X. Conversely, if Z is a subspace of X of deficiency 0 or 1, then there is an $f \in X^*$ such that $Z = Z(f)$. **(7+8)**

2. (a) Let X be a real normed linear space. Prove that for any $x \neq 0$ in a normed linear space X, there is an $x' \in X'$such that$x'(x) = \|x\|$and$\|x'\| = 1$. **(5)**

(OR)

(b) Let X and Y be normed linear spaces and T be a linear transformation. Prove that T is continuous if and only if T is bounded. **(5)**

(c) State and prove uniform boundedness principle theorem. **(15)**

(OR)

(d) State and prove Hahn Banach Theorem for a complex normed linear space. **(15)**

3. (a) Let X be a normed vector space and let X' be the dual space of X. If $X'$is separable then prove that X is separable. **(5)**

(OR)

(b) A Banach space X is either reflexive or its successive second dual space $X'', X''', \ldots$ are all distinct. How can we verify this statement? **(5)**

(c) (i) Define projection on a Banach space.

(ii) If P is a projection on a Banach space X and if M and N are its range and null space respectively then prove that M and N are closed linear subspaces of X such that $X = M \oplus N$.

(iii) If M is a direct sum of X and N is a closed subspace with $X = M \oplus N$ and with unique representation $x = y + z$ where $y \in M$, $z \in N$ then prove that P is a projection where $Px = y$.

(2+6+7)

(OR)

(d) State and prove open mapping theorem. **(15)**

4. (a) Prove that a real Banach space is a Hilbert space if and only if the parallelogram law holds.

(5)

(OR)

(b) State and prove Pythagorean theorem. **(5)**

(c) If M is a closed subspace of a Hilbert space X, then prove that every $x \in X$ has unique representation $x = y + z,\ y \in M, z \in M^{\perp}$. **(15)**

(OR)

(d) If $P_1, P_2, \ldots, P_n$ are the projections on a closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Hilbert space H, then prove that $P = P_1 + P_2 + \cdots + P_n$ is a projection if and only if P_i's are pairwise orthogonal. Also, show that P is a projection on $M = M_1 + M_2 + \cdots + M_n$.

(15)

5. (a) Let A be Banach algebra. Show that the inverse of the regular element $x \in A$ is $x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$.

(5)

(OR)

(b) Let A be Banach algebra. Let Z denote the set of all topological divisors of zero in A. Then prove that every zero divisor in A is a topological divisors of zero in A and Z is a subset of S.

(5)

(c) State and prove the Spectral theorem. **(15)**

(OR)

(d) Define spectrum of an element belonging to a Banach algebra. Prove that the spectrum of x, $\sigma(x)$ is non-empty. Also prove that the spectrum of x, $\sigma(x)$ is a compact subset of the complex plane.

(15)

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