LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – NOVEMBER 2016

16PMT1MC05 - PROBABILITY THEORY AND STOCHASTIC PROCESS

Date: 12-11-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer ALL Questions:

1. (a) The joint probability distribution of two random variables X and Y is given by: $P(X=0,Y=1)=\frac{1}{3}$, $P(X=1,Y=-1)=\frac{1}{3}$, and $P(X=1,Y=1)=\frac{1}{3}$. Find (i) Marginal distribution of X and Y, and (ii) the conditional probability distribution of X given Y=1.

OR

(b) Show that for t- distribution with n. degree of freedom , mean deviation about mean is given by $\sqrt{n}\Gamma\left[\frac{n-1}{2}\right]/\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)$. (5)

(c) The height of six randomly chosen sailors are (in inches): 63, 65, 68, 69, 71 and 72. Those of randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Discuss, the light that these data throw on the suggestion that sailors are on the average taller than soldiers. (8)

(d) If the random variables X_1 and X_2 are independent and follow chi-square distribution with p. d. f., show that $\sqrt{n}(X_1-X_2)/2\sqrt{X_1X_2}$ is distributed as student's t with n.d.f., independently of X_1+X_2. (7)

OR

(e) Given $f(x,y)=e^{-(x+y)}I_{(0,\infty)}(x).I_{(0,\infty)}(y)$. Are X and Y independent?. Find (i) $P(X>1)$, (ii)$P(X<Y\,/\,X<2Y)$, (iii) $P(1<X+Y<2)$. (7)

(f) Two random samples gave the following results:

Sample	Size	Sample mean	Sum of squares of deviations from the mean
1	10	15	90
2	12	14	108

Test whether the sample come from the normal population at 5% level of significance.
[$F_{0.05}(9,11)=2.90$, $F_{0.05}(11,9)=3.10$, $t_{0.05}(20)=2.086$, $t_{0.05}(22)=2.07$ (8)

2. (a) State and prove weak law of large number.

OR

(b) Two unbiased dice are thrown. If X is the sum of the numbers showing up, prove that $P(|X-7|\geq 3)\leq\frac{35}{54}$. (5)

(c) If the variables are uniformly bounded, then prove that the condition, $\lim_{n\to\infty}\frac{B_n}{n^2}=0$, is necessary as well as sufficient for WLLN to hold. (8)

(d) How large a sample must be taken in order that the probability will be at least 0.95 that $\overline{X_n}$ will be within 0.5 of μ. (7)

OR

(e) A random variable X assumes the values $\lambda_1,\lambda_2,\ldots$ with probabilities $u_1,u_2,\ldots$ respectively. Show that $p_k=\frac{1}{k!}\sum_{j=0}^{\infty}u_je^{-\lambda_j}\left(\lambda_j\right)^k;\lambda_j>0,\sum u_j=1$ is a probability distribution. Find its generating function and prove that its mean equals E(X) and variance equals V(X) + E(X). (7)

(f) State and prove the converse of Borel- Cantelli lemma. (8)

3. (a) If T_1 and T_2 are two unbiased estimators of $\gamma(\theta)$, having the same variance and ρ is the correlation between them, then show that $\rho \geq 2e - 1$, where e is the efficiency of each estimator.

OR

(b) Estimate α and β in the case of Pearson's Type III distribution by the method of moments: $f(x;\ \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, 0 \leq x < \infty$. (5)

(c) State and prove the invariance property of consistent estimator. (7)

(d) X_1, X_2 and X_3 is a random sample of size 3 from a population with mean value μ and variance σ^2. T_1, T_2, T_3 are the estimators used to estimate mean value μ, where $T_1 = X_1 + X_2 - X_3$, $T_2 = 2X_1 - 4X_2 + 3X_3$ and $T_3 = \frac{\lambda X_1 + X_2 + X_3}{3}$.

(i) Are T_1 and T_2 unbiased estimators?

(ii) Find the value of λ such that T_3 is unbiased estimator of μ.

(iii) With this value of λ is T_3 a consistent estimator?

(iv) Which is the best estimator? (8)

OR

(e) If T_1 and T_2 be two unbiased estimators of $\gamma(\theta)$ with variances $\sigma_1{}^2, \sigma_2{}^2$ and correlation ρ, what is the best unbiased linear combination of T_1 and T_2 and what is the variance of such a combination? (7)

(f) Find the M. L. E of the parameters α and λ(λ being large) of the distribution $f(x; \alpha, \lambda) = \frac{1}{\Gamma(\lambda)} \left(\frac{\lambda}{\alpha}\right)^{\lambda} e^{\frac{-\lambda x}{\alpha}} x^{\lambda-1}, 0 \leq x \leq \infty, \lambda > 0$, where $\frac{\partial}{\partial \lambda} \log \Gamma(\lambda) = \log \lambda - \frac{1}{2\lambda}$ and $\frac{\partial^2}{\partial \lambda^2} \log \Gamma(\lambda) = \frac{1}{\lambda} + \frac{1}{2\lambda^2}$. (8)

4. (a) Given the frequency function $f(x, \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & elsewhere \end{cases}$ and what you are testing the null hypothesis $H_0: \theta = 1$ against $H_1: \theta = 2$, by means of a single observed value of x. What would be the sizes of the type I and type II errors, if you choose the interval $0.5 \leq x$ as the critical region? Also obtain the power function of the test.

OR

(b) Write a short note on sign test. (5)

(c) State and prove Neyman Pearson Lemma (8)

(d) Use the Neyman-Pearson Lemma to obtain the region for testing $\theta = \theta_1 > \theta_0$ and $\theta = \theta_1 < \theta_0$, in the case of normal population $N(\theta, \sigma^2)$, where σ^2 is known. (7)

OR

(e) Prove that most powerful (MP) or uniformly (UMP) critical region (CR) is necessarily unbiased.

(i) If W be an Most Powerful Critical Region of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, when it is necessarily unbiased.

(ii) Similarly if W be Uniformly Most Powerful Critical Region of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta \in \Theta$, then it is also unbiased. (8)

(f) Write down the advantages and disadvantages of non-parametric tests. (7)

5. (a) A continuous random variable X has a p. d. f. $f(x) = 3x^2, 0 \leq x \leq 1$. Find a and b such that (i) $P(X \leq a) = P(X > a)$, and (ii) $P(X > b) = 0.05$

OR

(b) Write a short note on classification of stochastic process. (5)

(c) Briefly explain a time dependent general birth and death process in stochastic process. (15)

OR

(d) Let P be the transition probability matrix of a finite Markov chain with elements p_{ij} ($i, j =$ $0, 1, 2, \ldots, k-1$). Then prove that n-step transition probabilities $p_{ij}^{(n)}$ are then obtained as the elements of the matrix P^n. (8)

(e) Let $X \sim N(\mu, 4)$, μ unknown. To test $H_0: \mu = -1$ against $H_1: \mu = 1$, based on a sample of size 10 from this population, we use the critical region: $x_1 + 2x_2 + \cdots + 10x_{10} \geq 0$. What is its size? What is the power of the test? (7)