LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – NOVEMBER 2016

MT 2811 - MEASURE THEORY AND INTEGRATION

Date: 11-11-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer **ALL** questions:

1. (a) Let *f* and *g* be measurable functions on the set *E* then prove that $f+g$ and fg are measurable functions. **(5)**

(OR)

(b) Prove that the class M of Lebesgue measurable sets is a Sigma Algebra. **(5)**

(c) Prove that the outer measure of an interval equals to its length. **(15)**

(OR)

(d) (i) Prove that not every measurable set is a Borel set.

(ii) Prove that every interval is measurable. **(7+8)**

2. (a) Show that $\lim \int_0^\infty \frac{dx}{(1+\frac{x}{n})^n x^{1/n}} = 1.$ **(5)**

(OR)

(b) Show that $\int_0^\infty \frac{\sin t}{e^t - x}\,dt = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+1}, -1 \le x \le 1.$ **(5)**

(c) If *f* and *g* be integrable functions, then prove the following:

(i) af is integrable and $\int af\,dx = a\int f\,dx.$

(ii) $f+g$ is integrable, and $\int (f+g)\,dx = \int f\,dx + \int g\,dx$

(iii) If $f = 0\ a.e.$, then $\int f\,dx = 0.$

(iv) If $f \le g\ a.e.$, then $\int f\,dx \le \int g\,dx.$

(v) If A and B are disjoint measurable sets, then $\int_A f\,dx + \int_B f\,dx = \int_{A\cup B} f\,dx$ **(15)**

(OR)

(d) Let *f* be a bounded function defined on the finite interval $[a,b]$, then prove that f is Riemann integrable over $[a,b]$ if and only if it is continuous a.e. **(15)**

3. (a) Show that every algebra is a ring and every σ-algebra is a σ-ring. **(5)**

(OR)

(b) Define measure, outer measure and complete measure on a ring $\mathcal{R}$ and show that if $A, B \in \mathcal{R}$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$. **(5)**

(c) If μ is a σ-finite measure on a ring □, then prove that it has a unique extension to the σ-ring $S(\mathfrak{R})$. **(15)**

(OR)

(d) Define a complete measure. Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let S^* denote the class of μ^* - measurable sets. Prove that S^* is a σ - ring and μ^* restricted to S^* is a complete measure **(15)**

4. (a) Define a convex function and prove that for a convex function ψ on (a,b) such that

$a < s < t < u < b$, then $\psi(s,t) \leq \psi(s,u)$. **(5)**

(OR)

(b) State and prove Jensen's Inequality. **(5)**

(c) (i) State and prove Holder's Inequality.

(ii) Let ψ be a function on (a, b). Then prove that ψ is convex on (a, b) if and only if for each *x* and *y* such that $a < x < y < b$, the graph of ψ on (a, x) and (y, b) does not lie below the line passing through $(x, \psi(x))$ and $(y, \psi(y))$. **(9+6)**

(OR)

(d) State and Prove Minkowski's inequality. **(15)**

5. (a) Prove that the countable union of sets with respect to a signed measure v is a positive set.

(OR)

(b) Let v be a signed measure on [X, ***S***]. Then prove that there exists a positive set A and a negative set B such that $A \cup B = X$, $A \cap B = \Phi$. **(5)**

(c) State and prove Radon-Nikodym Theorem. **(15)**

(OR)

(d) State and prove Jordan decomposition theorem. **(15)**