LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc.DEGREE EXAMINATION – **MATHEMATICS**

FIRSTSEMESTER – NOVEMBER 2017

17PMT1MC02- REAL ANALYSIS

Date: 04-11-2017 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer all Questions. All questions carry equal marks.

1. (a) Suppose f is a continuous mapping of a metric space X into a metric space and E is a connected subset of X. Then prove that $f(E)$ is connected.

(OR)

(b) State and prove generalized mean value theorem. (5 marks)

(c) (i) Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point . $x \in [a,b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point f(x). If $h(t) = g(f(t)), a \le t \le b$, then prove that *h* is differentiable at *x* and $h'(x) = g'(f(x))f'(x)$. (9 marks)

(ii) If f is a real valued function defined on $[a,b]$, f has local maximum at a point $x \in [a,b]$ and $f'(x)$ exists, then prove that $f'(x) = 0$. (6 marks)

(OR)

(d) (i) Suppose f is a real differentiable function on [a,b] and suppose $f'(a) < \lambda < f'(b)$. Prove that there is a point x$\in (a,b)$ such that $f'(x) = \lambda$. (8 marks)

(ii) Suppose f is a continuous mapping of [0, 1] into itself. Prove that f(x) =x for atleast one $x \in [0,1]$. (7 marks)

2. (a) For $f(x) = 2x^2 + 1, \alpha(t) = t + \lfloor 3t \rfloor$ and P be the partition of $[0,1]$ consisting of four subintervals of equal length find $U(P,f,\alpha)$ and $L(P,f,\alpha)$.

(OR)

(b) If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on [a, b], then prove that $|f| \in \Re(\alpha)$ and $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$.

(5 marks)

(c) (i) State and prove the fundamental theorem of calculus (5 marks)

(ii) State and prove a necessary and sufficient condition for a bounded real valued function to be a Riemann-Steiltjes integrable. (10 marks)

(OR)

(d) (i) State and prove the theorem on Integration by parts.

(ii) If f is a real continuously differentiable function on [a,b] with f (a) = f (b) = 0 and $\int_a^b f^2(x)dx = 1$, then prove that $\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$ (5+10 marks)

3. (a)Prove that for $f_n(x) = n^2 x(1-x^2)^n, 0 \le x \le 1, n = 1,2 \ldots,$

$\int_0^1 (\lim_{n\to\infty} f_n(x))\, dx \ne \lim_{n\to\infty} \int_0^1 f_n\,(x)dx$.

(OR)

(b)State and prove the Cauchy criterion for uniform convergence of sequence of functions

(5 marks)

(c) If $\{f_n\}$ is a sequence of continuous functions on a set E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E. (15marks)

(OR)

(d)State and prove the Stone-Weierstrass theorem. (15 marks)

4. (a) Let= $\{\varphi_0, \varphi_1, \varphi_2, ...\}$ be orthnormal on I and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows: $s_n(x) = \sum_{k=0}^{\infty} c_k\varphi_k(x)$, $t_n(x) = \sum_{k=0}^{\infty} b_k\varphi_k(x)$ where $c_k = (f, \varphi_k(x)$ for k = 0, 1, 2… and b_0, b_1, b_2 … are arbitrary complex numbers. Then for each n, prove that$\|f - s_n\| \leq \|f - t_n\|$.

(OR)

(b)State and prove the Riesz-Fischer theorem. (5 marks)

(c) State and prove the Riemann-Lebesgue lemma and use the lemma to prove the following: For $f \in L(-\infty, +\infty)$, $\lim_{\propto\to\infty} \int_{-\infty}^{\infty} f(t)\frac{1-cos\propto t}{t}dt = \int_0^{\infty} \frac{f(t)-f(-t)}{t}dt$. (15 marks)

(OR)

(d) (i) If g is of bounded variation on$[0, \delta]$, then prove thatl $\text{im}_{\alpha\to\infty} \frac{2}{\pi}\int_0^{\delta} g(t)\frac{sin\alpha t}{t}dt = g(0+)$.

(ii) Suppose that $g(0+)$ exists and for some δ>0, the Lebesgue integral $\int_0^{\delta} \frac{g(t)-g(0+)}{t}dt$ exists. Prove that l $\text{im}_{\alpha\to\infty} \frac{2}{\pi}\int_0^{\delta} g(t)\frac{sin\alpha t}{t}dt = g(0+)$. (9+6 marks)

5. (a) If Ω is the set of all invertible linear operators on R^n and for $A \in \Omega, B \in L(R^n)$, if$\|B - A\|\|A^{-1}\| < 1$, then prove that $B \in \Omega$.

(OR)

(b) State and prove the fixed point theorem. (5 marks)

(c) State and prove the inverse function theorem.

(OR)

(d) State and prove the implicit function theorem. (15 marks)

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