Integration of Functions of Real Variable

Overview of the Chapter

- Simple Functions.
- Lebesgue Integral.
- Convergence theorem.
- Lebegue Integrable over Measurable Sets.
- Integration of Series.
- Riemann and Lebesgue Integrals.
- Integration of Complex Functions.

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

Lecture: February 17 & 19, 2025

Topic: Integration of non-negative functions: Simple functions – Lebesgue Integral, Fatou's Lemma.

Definition: For a set A, the characteristic function of set A (denoted by χ_A) is defined as

$$\chi_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \in A^c \end{cases}$$

Definition: A function φ defined on R is a simple function if

- $\varphi(x)$ is non-negative.
- $\varphi(x)$ is finite valued function
- $\varphi(x)$ takes only finite number of different values.

Try yourself:

1. Give examples for a simple function.

Notation: Let $a_1, a_2 \dots a_n$ are distinct values taken by $\varphi(x)$.

Define $A_i = \{x \in R : \varphi(x) = a_i,\ 1 \leq i \leq n\}$. These are subsets A_i of R where $\varphi(x)$ takes values $a_i,\ 1 \leq i \leq n$.

Properties of the sets A_i:

(i) $A_i \cap A_i = \emptyset\ for\ i \neq j$.

(ii) $\bigcup_{i=1}^{n} A_i = R$.

(iii) A_i's are measurable if φ is a measurable function.

Representation of a simple function φ:

If χ_{A_i} is a characteristic function of the set A_i, then $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x),\ \ for\ x \in R$.

i.e., a simple function $\varphi(x)$ can be written as a sum of weighted characteristic function.

Integral of a simple function φ:

Definition: Let φ be a measurable simple function which takes the value a_i on $A_i, 1 \leq i \leq n$. Then the integral of φ is defined as $\int \varphi dx = \sum_{i=1}^{n} a_i\, m(A_i)$ where a_i are distinct values taken by φ, A_i are measurable sets corresponding to each a_i, $m(A_i)$ is the Lebesgue measure of A_i.

Lebesgue integral for a non-negative function:

Definition 1: For any non-negative measurable function *f*, the Lebesgue integral of *f* is defined as

$\int f dx = sup\{\int \varphi dx\}$,

where supremum is taken over all measurable simple functions φ such that $\varphi \leq f$.

Integral over a set E:

Definition 2: For any measurable set *E* and any non-negative measurable function *f* defined on *E*,

$\int_E \; f dx = \int f \chi_E dx$ is the integral of *f* over *E*.

Note:

1. If $E = [a, b]$, then the integral is denoted by $\int_a^b f dx$.
2. If $a > b$, then $\int_a^b f dx = -\int_b^a f dx$.

Try yourself:

1. If φ is a measurable simple function, its integral can be found using Definitions 1 and 2. Verify that these values are identical.

Theorem: If φ is a measurable simple function, then

(i) $\int_E \; \varphi dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set *E*.

(ii) $\int_{A \cup B} \varphi dx \int_A \; \varphi dx + \int_B \; \varphi dx$ for any disjoint measurable sets, *A* and *B*.

(iii) $\int a\varphi dx = a \int \varphi dx$ if $a > 0$.

Proof:

(i) Given that *E* is a measurable set.

Define $\int_E \; \varphi dx = \int_E \; \varphi . \chi_E \, dx \rightarrow (1)$ where χ_E is the characteristic function of *E*.

If set $\{a_1, a_2 \ldots a_n\}$ is a set of non-zero values of φ, then $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x: \varphi(x) = a_i\}$ and $A_i \subseteq E$.

Since E is measurable set, its characteristic function χ_E is measurable. This implies that $\varphi.\chi_E$ is measurable and $\{x: \varphi.\chi_E(x) = a_i\} = A_i \cap E$.

Now $\int_E \varphi.\chi_E\, dx = \sum_{i=1}^{n} a_i\, m(A_i \cap E) \to (2)$

From (1) and (2), $\int_E \varphi dx = \sum_{i=1}^{n} a_i\, m(A_i \cap E)$.

(ii) **Additivity over disjoint sets:** Since $A\ and\ B$ are disjoint measurable sets, $m(A \cup B) = m(A) + m(B)$.

$\int_A \varphi dx + \int_B \varphi dx = \sum_{i=1}^{n} a_i\, m(A_i \cap A) + \sum_{i=1}^{n} a_i\, m(A_i \cap B)$ by (i)

$$= \sum_{i=1}^{n} a_i\,[m(A_i \cap A) + m(A_i \cap B)]$$

$$= \sum_{i=1}^{n} a_i\,[m(A \cup B) \cap A_i)]$$

$= \int_{A \cup B} \varphi\, dx$ by (i)

(iii) **Scaling by a positive constant:** For a constant $a > 0$, consider the set $\{x: a\varphi(x) = aa_i\} = A_i$ as $\{x: \varphi(x) = a_i\} = A_i$.

$\int a\varphi\, dx = \sum_{i=1}^{n} aa_i\, m(A_i) = a\sum_{i=1}^{n} a_i\, m(A_i) = a\int \varphi\, dx$.

Theorem: If f is non-negative measurable function, then $f = 0\ a.e.$ if and only if $\int f dx = 0$.

Proof: Assume that $f = 0\ a.e.$

Let φ be a measurable simple function with $\varphi = \{a_1, a_2 \dots a_n\}$ such that $\varphi \le f$.

Since $f = 0$ and $\varphi \le f$, $\varphi = 0\ a.e.$

$\Rightarrow a_i = 0$ except over a set of measure zero.

Thus $\int \varphi\, dx = \sum_{i=1}^{n} a_i\, m(A_i) = 0 \to (1)$

By definition of Lebesgue integral of non-negative function, $\int f\, dx = sup \int \varphi\, dx \to (2)$ for all $\varphi \le f$.

Using (1) and (2), $\int f\, dx = 0$.

Conversely, suppose that $\int f\, dx = 0$.

Define $E_n = \left\{x: f(x) > \frac{1}{n}\right\}$.

$\int f\, dx = \int_{E_n} f.\chi_{E_n} \ge \frac{1}{n}\int \chi_{E_n}\, dx \to (3)$

Since χ_{E_n} is a simple function taking values 1 and 0,

$$\int \chi_{E_n}\, dx = 1.m(E_n) + 0.m({E_n}^c) = m(E_n) \to (4)$$

Using (4) in (3), $\int_{E_n} f.\chi_{E_n} \geq \frac{1}{n} m(E_n)$.

Since $\int f\, dx = 0$, $\int_{E_n} f\, dx = 0$. This implies, $m(E_n) = 0$ for $n \geq 1$.

But $\{x: f(x) > 0\} = \cup_{n=1}^{\infty} E_n$.

Thus, $m(\{x: f(x) > 0\}) = \sum_{i=1}^{\infty} m(E_i) = 0$. Therefore, $f = 0\ a.e.$

Theorem: If f and g are non-negative measurable functions, then

(i) **Monotonic Property:** If $f \leq g$, then $\int f dx \leq \int g dx$.

(ii) **Monotonic over measurable sets:** If A is a measurable set and $f \leq g$ on A, then

$$\int_A f dx \leq \int_A g dx.$$

(iii) If $a \geq 0$, then $\int af dx = a \int f dx$

(iv) If A and B are measurable sets with $A \supseteq B$, then $\int_A f dx \geq \int_B f dx$.

Theorem (Fatou's Lemma): Let $\{f_n\}, n \geq 1$ be a sequence of non-negative measurable functions. Then $liminf \int f_n dx \geq \int liminf f_n dx$.

Question: Give an example where strict inequality occurs in Fatou's Lemma.

Question: Is non-negativity for the sequence $\{f_n\}$ of measurable functions is a necessary condition in Fatou's lemma.?

****

Refer:

1. G De Barra, Measure Theory and Integration, New Age International Publishers, Second Edition, 2019.

Lecture: February 24, 2025

Topic: Integration of non-negative functions: Lebesgue Monotone Convergence Theorem, Integrable Functions, Lebesgue Dominated Convergence Theorem.

Theorem (Lebesgue's Monotone Convergence Theorem): Let $\{f_n\}, n \geq 1$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotonically increasing for each x. and $\lim f_n = f$ $a.e.$. Then $\int f dx = lim \int f_n dx$.

Proof: Since $f_n \to f$, $f = \lim inf f_n = \lim sup f_n$.

Then $\int f dx = \int \lim inf f_n\, dx \leq \lim\inf \int f_n\, dx \to (1)$ [By theorem: Let $\{f_n\}, n \geq 1$ be a sequence of non-negative measurable functions. Then $liminf \int f_n dx \geq \int liminf f_n dx$].

Since $\{f_n(x)\}$ is monotonically increasing function with limit f, $f_n \leq f$.

$\Longrightarrow \int f_n\, dx \leq \int f\, dx$.

Since $\int f_n\, dx$ is bounded above by $\int f\, dx$, limit supremum exists and $\lim\sup \int f_n\, dx \leq \int f\, dx \to$ (2)

By equations (1) and (2), $\lim\sup \int f_n\, dx \leq \int f\, dx \leq \lim\inf \int f_n\, dx$.

$\Longrightarrow \int f\, dx = lim \int f_n dx$.

Question: Will Monotone convergence theorem hold good for a decreasing sequence of functions.

Theorem: For a measurable function $f \geq 0$, there exists a sequence $\{\psi_n\}$ of measurable simple functions such that $\psi_n(x) \uparrow f(x)$ for each x.

Theorem: Let f and g be non-negative measurable functions, then $\int f dx + \int g dx = \int (f+g) dx$.

Proof: Let $a_1, a_2 \dots a_n$ be values of φ taken on the sets $A_1, A_2 \dots A_n$ such that $\bigcup_{n=1}^{\infty} A_n = R$. Also, let $b_1, b_2 \dots b_n$ be values of ψ taken on the sets $B_1, B_2 \dots B_m$ such that $\bigcup_{m=1}^{\infty} B_m = R$. Then the simple measurable function $\varphi + \psi$ has values $a_i + b_j$ taken on the measurable set $A_i \cap B_j$.

Then $\int_{A_i \cap B_j} (\varphi + \psi)\, dx = \sum_{i=1}^{n} (a_i + b_j) m(A_i \cap B_j) = \int_{A_i \cap B_j} \varphi\, dx + \int_{A_i \cap B_j} \psi\, dx \to (1)$ [By theorem: **Theorem:** If φ is a measurable simple function, then $\int_E \varphi dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set E.]

Also, $\bigcup_{i,j} (A_i \cap B_j) = \bigcup_i \bigcup_j (A_i \cap B_j) = \bigcup_j (\bigcup_i A_i \cap B_j) = \bigcup_j (R \cap B_j) = \bigcup_j B_j = R$. This implies that the union of these mn sets $A_i \cap B_j$ is R. So, applying theorem [**Theorem:** If φ is a measurable simple function, then $\int_{A\cup B} \varphi dx \int_A \varphi dx + \int_B \varphi dx$ for any disjoint measurable sets, A and B.] to both sides of equation (1) we have,

$$\int (\varphi + \psi) dx = \int \varphi\, dx + \int \psi\, dx \rightarrow (2)$$

Let f and g be non-negative measurable functions and $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of measurable simple functions such that $\varphi_n \uparrow f$ and $\psi_n \uparrow g$. Then $\{\varphi_n + \psi_n\}$ is an increasing sequence of measurable simple functions such that $\varphi_n + \psi_n \uparrow f + g$.

By equation (2), $\int (\varphi_n + \psi_n) dx = \int \varphi_n\, dx + \int \psi_n\, dx$.

Using Lebesgue Monotone Convergence theorem and making $n \rightarrow \infty$. we have $\int f dx + \int g dx = \int (f + g) dx$.

Theorem: Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then $\int \sum_{n=1}^{\infty} f_n\, dx = \sum_{n=1}^{\infty} \int f_n\, dx$.

Try yourself:

1. Let $f, g \geq 0$ be measurable functions with $f \geq g$ and $\int g dx < \infty$. Show that $\int f dx - \int g dx = \int (f - g) dx$.
2. Let $f(x) = 0$ at each point $x \in P$ where P is the Cantor set in $[0,1]$ and $f(x) = p$ in each complementary interval of length 3^{-p}. Show that f is measurable and that $\int_0^1 f\, dx = 3$.
3. The function f is defined on $(0,1)$ by
$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ [1/x]^{-1} & x \text{ is irrational} \end{cases}$$
where $[1/x]$ is the integral part of x. Show that $\int_0^1 f\ dx = \infty$.
4. Statement 1: Let $\{f_n\}, n \geq 1$ be a sequence of non-negative measurable functions. Then $\lim\ inf \int f_n dx \geq \int \lim\ inf f_n\, dx$.

   Statement 2: If $\{f_n\}, n \geq 1$ is a sequence of non-negative measurable functions and $\lim\ f_n = f\ a.e.$, then $\int f\, dx \leq \int \lim\ inf f_n\, dx$.

   Show that Statement 1 and 2 are equivalent.
5. Let $f_n(x) = \min(f(x), n)$, $f \geq 0$ where f is measurable. Show that $\int f_n dx \uparrow \int f dx$.

****

GENERAL INTEGRAL

Definition: If $f(x)$ is any real function, then $f^+ = \max(f(x), 0)$, $f^- = max(-f(x), 0)$ are said to be the positive and negative parts of f respectively.

Theorem:

(i) $f = f^+ - f^-$.

(ii) $|f| = f^+ + f^-$.

(iii) $f^+ \geq 0 \; and \; f^- \geq 0$.

(iv) f is measurable if and only if $f^+ and \; f^-$ are both measurable.

Proof: For any two functions f and g, $\max(f, g) = \frac{|f-g| + f + g}{2}$.

Substituting $g = 0$, $\max(f, 0) = \frac{|f| + f}{2}$.

Then $f^+ = \max(f(x), 0) = \frac{|f| + f}{2}$, $f^- = max(-f(x), 0) = \frac{|f| - f}{2}$.

$\Rightarrow f = f^+ - f^-$ and $|f| = f^+ + f^-$

Proving (i) and(ii)

Obviously $f^+ \geq 0 \; and \; f^- \geq 0$.

Let f be measurable. Then $\max(f(x), 0)$, $max(-f(x), 0)$ are measurable functions.

$\Rightarrow f^+, f^-$ are measurable functions.

Conversely, if f^+, f^- are measurable functions then $f^+ - f^-$ is measurable which implies f be measurable.

Question: Find the positive and negative parts of f in [– 2, 2] where $f(x) = x^2 - 1$.

Question: Find the positive and negative parts of f in [– 3, 3] where $f(x) = e^x - 4$.

Question: Find the positive and negative parts of f in [– 5, 5] where

$$f(x) = \frac{x^2 - 5}{x+5}, exclude \; the \; point \; x = -5.$$

Question: Determine f^+, f^- for $f(x) = \frac{1}{2} + sinx$ where $0 \leq x \leq 2\pi$.

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

Definition: If f is measurable and $\int f^+ \, dx < \infty$, $\int f^- \, dx < \infty$, then f is integrable and its integral is given by $\int f \, dx = \int f^+ dx - \int f^- \, dx$.

Theorem: A measurable function is integrable if and only if $|f|$ and $|f| = \int f^+ dx + \int f^- \, dx$.

Definition: If E is a measurable set, f is a measurable function and $\chi_E f$ is integrable, then f is integrable over E and its integral is given by $\int_E \, f \, dx = \int f \chi_E dx$. This is denoted by $f \in L(E)$.

Definition: If f is measurable function such that at least one of $\int f^+ \, dx$, $\int f^- \, dx$ is finite, then $\int f \, dx = \int f^+ dx - \int f^- \, dx$.

Theorem: Let f be an integrable function. Then af is integrable and $\int af \, dx = a \int f dx$.

Proof: Since f is an integrable function, f is a measurable. So, af is measurable for any $a \in R$.

Case ($\boldsymbol{a} \geq \mathbf{0}$): For $f > 0$,

$(af)^+ = \max(af, 0) = af = amax(f, 0) = af^+$.

Similarly, $(af)^- = af^-$.

$\int (af)^+ dx = \int af^+ \, dx = a \int f^+ \, dx$.

Since $\int f^+ \, dx < \infty$, $a \int f^+ \, dx < \infty$

$\Rightarrow \int (af)^+ dx < \infty$.

$\int (af)^- dx = \int af^- \, dx = a \int f^- \, dx$.

Since $\int f^- \, dx < \infty$, $a \int f^- \, dx < \infty$

$\Rightarrow \int (af)^- dx < \infty$.

Thus, af is integrable.

Also, $\int af \, dx = \int (af)^+ dx - \int (af)^- dx$ (by definition)

$= \int af^+ \, dx - \int af^- \, dx$

$= a[\int f^+ \, dx - \int f^+ \, dx]$

$= a \int f dx$

Case ($\boldsymbol{a} = -\mathbf{1}$):

$(af)^+ = (-f)^+ = \max(-f, 0) = f^-$.

$(af)^- = (-f)^- = \max(f, 0) = f^+$.

$\int (-f)^+ \, dx = \int f^- \, dx$. As $\int f^- \, dx < \infty$, $\int (-f)^+ \, dx < \infty$.

$\int(-f)^- \, dx = \int f^+ \, dx$. As $\int f^+ \, dx < \infty$, $\int(-f)^- \, dx < \infty$.

Thus, $-f$ is integrable.

$\int(-f) \, dx = \int(-f)^+ dx - \int(-f)^- dx = \int f^- \, dx - \int f^+ \, dx = -\int f dx.$

Case ($a < 0$): In this case, $|a| = -a$.

Then $af = -|a|f$.

Since f is an integrable function, $|a|f$ is an integrable function by Case ($a \geq 0$).

$\Rightarrow -|a|f$ is an integrable function by Case ($a = -1$).

Thus, af is integrable.

Also, $\int af \, dx = \int -|a|f \, dx = -|a| \int f \, dx = a \int f \, dx$.

Theorem: Let f and g are integrable functions. Then $f + g$ is integrable and it integral is given by

$$\int(f + g) \, dx = \int f dx + \int g dx.$$

Theorem: Let f and g are integrable functions.

(i) If $f = 0 \; a.e.$ then $\int f dx = 0$.

(ii) If $f \leq g \; a.e.$ then $\int f dx \leq \int g dx$.

Proof:

(i) Since $f = f^+ - f^-$ and $f = 0 \; a.e.$, $f^+ = 0 \; a.e.$ and $f^- = 0 \; a.e.$

$\int f^+ \, dx = 0$ and $\int f^- \, dx = 0$

[By theorem: If f is non-negative measurable function, then $f = 0 \; a.e.$ if and only if $\int f dx = 0$.]

Thus $\int f dx = \int f^+ \, dx - \int f^- \, dx = 0$.

(ii) As $g = f + (g - f)$, $\int g \, dx = \int\big(f + (g - f)\big) \, dx$

$$= \int f dx + \int (g - f)^+ \, dx - \int (g - f)^- \, dx \rightarrow (1)$$

Since $f \leq g \; a.e.$, $(g - f) \geq 0$.

$\Rightarrow (g - f)^+$ exists and $(g - f)^- = 0 \rightarrow (2)$

Using (2) in (1), $\int g \, dx = \int f dx + \int (g - f)^+ \, dx$.

Since $\int(g - f)^+ \, dx > 0$, $\int f dx \leq \int g dx$.

Theorem: Let f be an integrable function and the sets A, B are disjoint measurable sets then

$$\int_A f dx + \int_B f dx = \int_{A \cup B} f dx.$$

Proof: We know that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.

Since $A \cap B = \emptyset$, $\chi_{A \cap B} = 0$. So that $\chi_{A \cup B} = \chi_A + \chi_B$.

$$\int_{A\cup B} f\,dx = \int f\,.\chi_{A\cup B}dx$$

$$= \int f.[\chi_A + \chi_B]\,dx$$

$$= \int f\,.\chi_A\,dx + \int f\,.\chi_B dx$$

$$= \int_A f\,dx + \int_B f\,dx.$$

Theorem: If f and g are measurable function, $|f| \le |g|\ a.e.$, and g is integrable, then f is integrable.

Proof: Since $|f| \le |g|\ a.e.$, $f^+ \le |g|$ and $f^- \le |g|$.

Since g is integrable, $|g|$ is integrable.

$\Longrightarrow \int|g|\,dx < \infty$.

As $\int f^+\,dx \le \int|g|\,dx$ and $\int f^-\,dx \le \int|g|\,dx$, $\int f^+\,dx < \infty$ and $\int f^-\,dx < \infty$.

Thus, the measurable function f is integrable.

Theorem: If f is an integrable function, then $|\int f dx| \le \int|f|dx$. Establish a necessary and sufficient condition for equality.

Theorem: If f is an integrable function, then f is finite-valued a.e.

Proof: Suppose f is not finite-valued a.e..

Then there is a set E with $m(E) > 0$ and $|f| = \infty$ on E.

$\Longrightarrow |f(x)| > n$ for large n and $x \in E$.

$\int_E |f|dx = \int|f|.\chi_E dx > n\int \chi_E\,dx = n\,m(E)$ for all n.

Since $m(E) > 0$, $\int_E |f|dx$ is not finite which is a contradiction. Thus f is finite-valued a.e..

Theorem: If f is measurable on measurable set E with $m(E) < \infty$. Suppose that there exist constants A, B such that $A \le f \le B$ on E, then $Am(E) \le \int_E fdx \le Bm(E)$.

Proof: Define constant functions g, h on E by $g(x) = A, h(x) = B$ for all $x \in E$.

Since $A \le f \le B$ on E, $g(x) \le f(x) \le h(x)$ on for all $x \in E$.

$\Longrightarrow \int_E g(x)\,dx \le \int_E f(x)\,dx \le \int_E h(x)\,dx$ [By theorem

$$\Longrightarrow A\int_E dx \le \int_E f(x)\,dx \le B\int_E dx$$

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

$\Longrightarrow Am(E) \leq \int_E f(x)\,dx \leq Bm(E).$

Theorem (Lebesgue's Dominated Convergence Theorem): Let $\{f_n\}$ be a sequence of measurable functions such that

(i) $|f_n| \leq g$, where g is integrable.

(ii) $lim f_n = f\ a.e.$

Then f is integrable and $lim \int f_n dx = \int f dx$.

Corollary: Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable and let $lim f_n = f\ a.e.$ Then $lim \int |f_n - f|\, dx = 0$.

Proof: Since $f_n \to f\ a.e.$, $|f_n - f| \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} |f_n - f| = 0$.

By Lebesgue's Dominated Convergence Theorem, $lim \int |f_n - f| dx = \int lim |f_n - f| dx = 0$.

Theorem: Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| dx < \infty$. Then

(i) the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e.

(ii) its sum $f(x)$ is integrable

(iii) $\int f dx = \sum_{n=1}^{\infty} \int f_n dx$.

Try yourself:

1. If f is measurable and g is integrable and α, β are real numbers such that $\alpha \leq f \leq \beta\ a.e.$, prove that there exists γ, $\alpha \leq \gamma \leq \beta$ such that $\int f|g| dx = \gamma \int |g| dx$.
2. Show that $\lim_{n\to\infty} \int_0^{\infty} \frac{dx}{\left(1+\frac{x}{n}\right)^n x^{1/n}} = 1$ using Lebesgue dominated convergence theorem.
3. Show that if $\alpha > 1$, $\int_0^1 \frac{x sinx}{1+(nx)^{\alpha}} dx = O(n^{-1})$ as $n \to \infty$.
4. Show that $\lim_{n\to\infty} \int_0^1 \frac{nx logx}{1+n^2x^2} dx = 0$ using Lebesgue dominated convergence theorem.
5. Evaluate $\lim_{n\to\infty} \int_0^1 f_n(x) dx$ where (i) $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$ (ii) $f_n(x) = \frac{nx}{1+n^2x^2}$ when $n = 0, 1, 2 \ldots$, $0 \leq x \leq 1$ using Lebesgue dominated convergence theorem.
6. Verify whether the function $f : [0,1] \to R$ defined by

$$f(x) = \begin{cases} \frac{1}{x^{2/3}}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is Lebesgue integrable.

7. Define $f:[0,1] \to R$ by $f(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$.

Show that f is not Lebesgue integrable.

8. Prove that the function f is not Lebesgue integrable where the function $f:[0,\infty] \to R$ defined by $f(x) = \begin{cases} \frac{sinx}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

9. Define $f:[-4,4] \to R$ by

$$f(x) = \begin{cases} 1, & when\ x\ is\ a\ irrational\ number\ in\ [-4,4] \\ 2, & when\ x\ is\ a\ rational\ number\ in\ [-4,4] \end{cases}$$

Find $\int_{-4}^{4} f(x)dx$.

10. Find the Lebesgue integral of the function

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & \{1 \leq x < 2\} \cup \{3 \leq x < 4\} \\ 2, & \{2 \leq x < 3\} \cup \{4 \leq x \leq 5\} \end{cases}$$

11. Show that the following functions are not Lebesgue integral:

(i) $f(x) = \frac{sinx}{x}$ defined on $(0,\infty)$ (ii) $f(x) = sinx + cosx$ defined in R.

12. Evaluate $\lim_{n\to\infty} \int_a^\infty \frac{n^2xe^{-n^2x^2}}{1+x^2}dx$, for $a > 0$, but not for $a = 0$.

13. Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then prove that $\int \sum_{n=1}^{\infty} f_n\, dx = \sum_{n=1}^{\infty} \int f_n dx$.

****

Refer:

G De Barra, Measure Theory and Integration, New Age International Publishers, Second Edition, 2019.

Lecture: February 25, 2025

Topics: Integration of non-negative functions: Integration of Series, Riemann and Lebesgue integrability.

INTEGRATION OF SERIES

Try yourself:

1. Show that $\int_0^1 \frac{x^{1/3}}{1-x}\log\frac{1}{x}\,dx = 9\sum_{n=1}^{\infty}\frac{1}{(3n+1)^2}$.
2. Evaluate $\int_0^1 sinxlogx\,dx$
3. Show that $\int_0^1 \left(\frac{logx}{1-x}\right)^2 dx = \frac{\pi^2}{3}$.
4. Show that $\int_0^{\infty} \frac{sint}{e^t-x}dx = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2+1}, -1 \le x \le 1$.
5. Show that $\int_0^{\infty} e^{-x}cos\sqrt{x}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{n!}{(2n)!}$.
6. Show that $\int_0^{\infty} \frac{x^{\alpha-1}}{e^x-1}dx = \left(\int_0^{\infty} x^{\alpha-1}e^{-x}dx\right)\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$.
7. Show that $\int_0^1 \frac{x^{p-1}}{1+x^q}dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{p+nq}$ when $p, q > 0$.

****

RIEMANN INTEGRALS

Recap:

Let $[a, b]$ be a given interval. A partition D of $[a, b]$ is a finite set of points $x_0, x_1 \dots x_n$ where $a = x_0 \le x_1 \le \cdots \le x_n = b$.

Define $\Delta x_i = x_i - x_{i-1}, 1 \le i \le n$.

Suppose f is a bounded function defined on $[a, b]$. Corresponding to each partition D of $[a, b]$, we have

$$M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$$

$$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$$

Riemann upper sum = $S_D = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$.

Riemann lower sum = $s_D = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$.

Upper Riemann integral of f on $[a, b]$ = $\mathcal{R}\int_{\bar{a}}^{b} f(x)dx$

$$= \sup\{s_D \mid \sup \textit{ is taken over all partitions } D \textit{ of } [a, b]\}$$

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

Lower Riemann integral of f on $[a,b] = \mathcal{R}\int_{a}^{\bar{b}} f(x)dx$

$= \inf\{S_D |\inf is\ taken\ over\ all\ partitions\ D\ of\ [a,b]\}$

If the upper and lower Riemann integral of f on $[a,b]$ are equal, then f is Riemann integrable on $[a,b]$ and the common value is denoted by $\mathcal{R}\int_a^b f dx$ and $\mathcal{R}\int_a^b f dx$ is the Riemann integral of f on $[a,b]$.

Note: Since f is bounded on $[a,b]$, there exist two numbers m, M such that $m \leq f(x) \leq M$ for $a \leq f(x) \leq b$.

Hence for every partition D, $m(b-a) \leq s_D \leq S_D \leq M(b-a)$. This implies that the upper and lower integrals are defined for every bounded function f.

Theorem: A bounded function f is Riemann integrable on $[a,b]$ if and only if for each $\epsilon > 0$, there exists a partition D of $[a,b]$ such that $S_D - s_D < \epsilon$.

Problems to solve:

1. Let $f(x) = x$ on $[0,1]$ and $D = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0,1]$. Find s_D, S_D.
2. If f is defined on $[0,1]$ by $f(x) = x$ for all $x \in [0,1]$, then prove that f is $\mathcal{R}$-integrable and $\int_0^1 f(x)dx = \frac{1}{2}$.
3. Define $f: [a,b] \to R$ by $f(x) = c_i, x_{i-1} < x < x_i$. Prove that f is Riemann integrable.
4. Show that $f(x) = 2x + 1$ is Riemann integrable on $[1,2]$ and $\int_1^2 f(x)dx = 4$.
5. Show by example that every bounded function need not be Riemann integrable.

****

Lecture: February 28, 2025

RIEMANN AND LEBESGUE INTEGRALS

Let *f* be a bounded real-valued function defined on the finite interval [a, b] and let $a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b$ be a partition D of $[a, b]$.

Then for each subdivision, define the sums $S_D = \sum_{i=1}^{n} M_i(\xi_i - \xi_{i-1})$, $s_D = \sum_{i=1}^{n} m_i(\xi_i - \xi_{i-1})$ where $M_i = sup_{\xi_{i-1} \le x \le \xi_i} f(x) \text{ and } m_i = inf_{\xi_{i-1} \le x \le \xi_i} f(x)$.

Theorem: If *f* is Riemann integrable and bounded over the finite interval $[a, b]$, then *f* is integrable and $\mathcal{R} \int_a^b f dx = \int_a^b f dx$.

Proof: Let $\{D_n\}$ be a sequence of partitions.

Since *f* is Riemann integrable, for each $n > 0$ there exists a partition D_n such that

$$S_D - s_D < \frac{1}{n} \rightarrow (1).$$

Choose step functions u_n and l_n depending upon the n^{th} partition. i.e., Define $u_n = M_i$ on $[\xi_{i-1}, \xi_i]$ and u_n be the average values of M_i corresponding to intervals ending at that point. So that $\int_{\xi_{i-1}}^{\xi_i} u_n \, dx = M_i \, l([\xi_{i-1}, \xi_i])$.

Then $\int_a^b u_n \, dx = \sum_{i=1}^{n} M_i \, l([\xi_{i-1}, \xi_i]) = S_{D_n}$.

Similarly, taking $l_n = m_i$ on $[\xi_{i-1}, \xi_i]$ and l_n be the average values of M_i corresponding to intervals ending at that point. So that $\int_{\xi_{i-1}}^{\xi_i} l_n \, dx = m_i \, l([\xi_{i-1}, \xi_i])$.

Then $\int_a^b l_n \, dx = \sum_{i=1}^{n} m_i \, l([\xi_{i-1}, \xi_i]) = s_{D_n}$.

Since u_n and l_n are step functions on $[a, b]$, u_n and l_n are simple functions on $[a, b]$. Since every simple functions are measurable, u_n and l_n are measurable functions on $[a, b]$.

By definition of u_n and l_n, $l_n \le f \le u_n$.

Define $U = \inf u_n$ and $L = \sup l_n$.

Since *f* is Riemann integrable, U, L exists and $U - L < \frac{1}{n}$ for large n.

The set $\{x: U(x) - L(x) > 0\} = \bigcup_{k=1}^{\infty} \left\{x: U(x) - L(x) > \frac{1}{k}\right\}$.

Since $U - L > \frac{1}{k}$, $u_n - l_n > \frac{1}{k}$ for each n.

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

If $m(\{x: U(x) - L(x) > 0\}) = A$, then $\int (u_n - l_n)dx > \frac{A}{k} \to (2)$

When n becomes large, $S_{D_n} - s_{D_n} = \int u_n\, dx - \int l_n\, dx = \int (u_n - l_n)dx$.

Since $S_{D_n} - s_{D_n} < \frac{1}{n}$, $\int (u_n - l_n)dx < \frac{1}{n} \to (3)$

By (1) and (2), $\frac{A}{k} < \int (u_n - l_n)dx < \frac{1}{n}$ for each n.

As n becomes large, $A = 0$. i.e., $m(\{x: U(x) - L(x) > 0\}) = 0$.

$\Rightarrow U - L \leq \frac{1}{k}\ a.e.$ for each k.

Since k is arbitrary, $U = L\ a.e.$

Since u_n and l_n are measurable, U and L are measurable.

Also, $L \leq f \leq U$.

$\Rightarrow f$ is measurable and bounded. Thus f is integrable.

Since $l_n \leq f \leq u_n$ and l_n, u_n are measurable, $\int_a^b l_n dx \leq \int_a^b f dx \leq \int_a^b u_n\, dx$.

As $n \to \infty$, $\mathcal{R} \int_{\bar{a}}^b f(x)dx \leq \int_a^b f(x)dx \leq \mathcal{R} \int_a^{\bar{b}} f(x)dx$.

Since f is Riemann integrable on $[a, b]$, $\mathcal{R} \int_a^b f(x)dx = \int_a^b f(x)dx$.

Theorem: If *f* is a bounded function defined on the finite interval [*a*, *b*], then *f* is Riemann integrable over [*a*, *b*] if and only if *f* is continuous a.e.

Definition: If for each *a* and *b*, *f* is bounded and Riemann integrable on [*a*, *b*] and $\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f dx$ exists, then *f* is said to be Riemann integrable on $(-\infty, \infty)$, and the integral is written as $\mathcal{R} \int_{-\infty}^{\infty} f dx$.

Question: Define $f: [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \notin Q \end{cases}$$

Show that *f* is Lebesgue integrable but not Riemann integrable.

Question: Show that *f* is Lebesgue integrable but not Riemann integrable for the function *f* defined by

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & (1 \leq x \leq 2) \cup (3 \leq x \leq 4) \\ 2 & (2 \leq x \leq 3) \cup (4 \leq x \leq 5) \end{cases}$$

Question: Find the Lebesgue integral of the function *f* defined by

$$f(x) = \begin{cases} x^2, & x \in [0,1] - Q \\ 1, & x \in [0,1] \cap Q \end{cases}$$

Is the function *f* Riemann integral on [0, 1]. Justify your answer.

Theorem: If *f* is a bounded function and $|f|$ be Riemann integrable on $(-\infty, \infty)$, then *f* is integrable and $\int_{-\infty}^{\infty} f\, dx = \mathcal{R}\int_{-\infty}^{\infty} f dx$.

Theorem: Let *f* be bounded and measurable on a finite interval [*a*, *b*] and $\epsilon > 0$. Then there exists

i. a step function *h* such that $\int_a^b |f - h| dx < \epsilon$.

ii. a continuous function *g* such that *g* vanishes outside a finite interval and $\int_a^b |f - g| dx < \epsilon$.

****

Refer:

1. G De Barra, Measure Theory and Integration, New Age International Publishers, Second Edition, 2019.

Prepared and posted by:

Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.

Topics: Integration of Complex Functions.

INTEGRATION OF COMPLEX FUNCTIONS

Suppose u and v are real-valued functions defined on a measure space $[\![X, \mathcal{S}, \mu]\!]$. Define a function $f: X \to R^2(= C)$ by $f(x) = (u(x), v(x)) = u(x) + iv(x)$ for each $x \in X$. Then the function $f = u + iv$ is a complex-valued function on $[\![X, \mathcal{S}, \mu]\!]$.(where u, v ($u: X \to R, v : X \to R$) are real and imaginary parts of f respectively).

A complex-valued function $f(= u + iv)$ defined on a measure space $[\![X, \mathcal{S}, \mu]\!]$ is Lebesgue measurable if and only if u and v are measurable. f is integrable if and only if u and v are integrable. If f is integrable, then define $\int f\, d\mu = \int u\, d\mu + i \int v\, d\mu$.

Theorem: For every complex measurable function f defined on X and $k \in C$, kf are measurable.

Theorem: For every complex measurable functions f, g defined on X, $f + g, fg$ and $|f|$ are measurable.

Proof: Let $f = u + iv$ and $g = u' + iv'$ where u, v, u', v' are real-valued measurable functions on X.

$f + g = (u + iv) + (u' + iv') = (u + u') + i(v + v')$.

Since u, v, u', v' are measurable functions, $u + u'$ and $v + v'$ are measurable.

[If f, g are extended be real-valued measurable functions, then $f + g$ is measurable.]

$\Rightarrow f + g$ is measurable.

$fg = (u + iv)(u' + iv') = (uu' - vv') + i(uv' + u'v)$.

Since u, v, u', v' are measurable functions, $uu' - vv'$ and $uv' + u'v$ are measurable.

[If f, g are extended be real-valued measurable functions, then $f + g, f - g$ are measurable.]

$\Rightarrow fg$ is measurable.

$|f|(x) = |f(x)| = |u(x) + iv(x)| = (u(x)^2 + v(x)^2)^{1/2}$.

Define a map $g: C \to R$ by $g(z) = |z|$ where $z = x + iy = (x, y)$. We find that map g is continuous.

$(gof)(x) = g(f(x)) = g(u(x) + iv(x)) = ((u(x))^2 + (v(x))^2)^{1/2} = |f|(x)$.

$\Rightarrow |f| = gof$.

Since f is measurable and g is continuous, gof is measurable.

[Let f be a continuous function and g be a measurable function. Then fog is measurable.]

$\Rightarrow |f|$ is measurable.

Theorem: A complex-valued measurable function f, $|\int f\,d\mu| \leq \int |f|\,d\mu$ whenever f is integrable.

Proof: We have $\int f\,d\mu = |\int f\,d\mu| e^{i\theta}$, for a suitable θ.

$$\Rightarrow \left|\int f\,d\mu\right| = e^{-i\theta}\int f\,d\mu$$

$$= \int Real\ part\ of\ \left(f e^{-i\theta}\right) d\mu$$

$$\leq \int \left|Real\ part\ of\left(f e^{-i\theta}\right)\right| d\mu$$

$$\leq \int |f|\,d\mu.$$

Definition: Suppose μ is a measure on X, E is a measurable subset of X and f is a complex-valued function on X. $f \in L(\mu)$ on E or f is integrable over E if f is measurable and $\int_E |f| d\mu < \infty$.

Theorem (Lebesgue's Dominated Convergence Theorem): Let $\{f_n\}$ be a sequence of complex-valued measurable functions such that $|f_n| \leq g\ pointwise\ a.e.$, where g is integrable extended real-valued function and let $f_n \to f\ pointwise\ a.e.$ Then f is integrable and $lim \int f_n dx = \int f dx$.

****

Refer:

2. G De Barra, Measure Theory and Integration, New Age International Publishers, Second Edition, 2019.
3. Walter Rudin, Principles of Mathematical Analysis, McGraw Hill Education (India) Private Ltd., 2013.

Prepared and posted by:
Dr. Chris Monica M., Assistant Professor, Department of Mathematics, Loyola College, Chennai 600 034.