LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc., DEGREE EXAMINATION - **MATHEMATICS**

SECOND SEMESTER – APRIL 2013

MT 2810/MT 2804 – ALGEBRA

Date: 26-04-2013 Dept. No. Max. : 100 Marks
Time: 9.00 – 12.00

Answer **ALL** the Questions:

1. a) Define the Normalizer of $a \in G$ and prove that N(a) is a subgroup of G.

(OR) (5)

b) Prove that if $O(G) = p^n$ where p is a prime number then $Z \neq (e)$ or $O(Z) > 1$.

c) If p is a prime number and p^α divides O(G) then prove that G has a subgroup of order p^α.

(OR) (15)

d) Show that the number of p-sylow subgroups in a finite group G is 1 + kp where p is a prime number. Also prove that any group of order 72 cannot be simple.

2. a) Given two polynomials $f(x)$, $g(x) \neq 0$ in F[x] then prove that there exists two polynomials t(x), r(x) in F[x] such that $f(x) = t(x)g(x) + r(x)$ where $r(x) = 0$ (or) deg $r(x) < \deg g(x)$.

(OR) (5)

b) If $f(x)$ and $g(x)$ are primitive polynomials then $f(x)g(x)$ is also a primitive polynomial.

c) Let R be an Euclidean Ring and M be the finitely generated R-module. Prove that M is the direct sum of a finite number of cyclic sub-modules.

(OR) (15)

d) State and Prove Eisenstein Criterion.

e) State and prove Gauss Lemma. (8+7)

3. a) If L is a finite extension of K and K is a finite extension of F then prove that L is a finite extension of F.

(OR) (5)

b) If R is a Unique Factorization Domain then prove that R[x] is a UFD.

c) Prove that the element $a \in K$ is algebraic over F iff F(a) is a finite extension of F.

(OR) (15)

d) If F is of characteristic zero and a, b are algebraic over F then prove that there exists an element $c \in F(a, b)$ such that $F(c) = F(a, b)$.

4. a) Prove that $\sqrt{3}$ *and* $\sqrt{5}$ are algebraic over Q. Find the degree of $\sqrt{3}+\sqrt{5}$ over Q and the basis of $Q\left(\sqrt{3},\sqrt{5}\right)$ over Q.

(OR) (5)

b) Prove that K is a normal extension of F iff K is the splitting field of some polynomial over F.

c) State and prove the fundamental theorem of Galois Theory.

(OR) (15)

d) Prove that S_n is not solvable for $n \geq 5$. (8+7)

e) Verify S_3 is solvable.

5. a) For every prime number p and for every positive integer m, prove that there is a unique field having p^m elements.

(OR) (5)

b) If F is a field, and α, β ≠ 0 are two elements of R then prove that we can find elements a, b in F such that $1+\alpha a^2+\beta b^2=0$.

c) State and prove Wedderburn's theorem on finite division rings.

(OR) (15)

d) If p(x) is a polynomial in F[x] of degree $n\geq 1$ and it is irreducible over F then there is an extension E of F such that $[E:F]=n$ in which p (x) has a root.

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