LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2018**

17/18PMT1MC01– LINEAR ALGEBRA

Date: 25-10-2018 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

I. a. (i) Prove that the similar matrices have the same characteristic polynomial.

(OR) **(5)**

(ii)) Let $A=\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ be the matrix of a linear operator T defined on R^3 with respect to the standard ordered basis. Prove that A is diagonalizable.

b. (i) Let T be a linear operator on a finite dimensional space V and $c_1, \dots c_k$ be the distinct characteristic values of T. Let W_i be the null space of $(T - c_iI)$. Prove that the following are equivalent.

(i) T is diagonalizable

(ii) The characteristic polynomial for T is $f=(x-c_1)^{d_1}\dots(x-c_k)^{d_k}$ and dim $W_i = d_i, i = 1,\dots,k.$

(iii) dim W_1 + … dim W_k = dim V.

(OR) **(15)**

(ii)State and prove Cayley – Hamilton Theorem.

II. a. (i) Let W be an invariant subspace for T. Then prove that the minimal polynomial for T_w divides the minimal polynomial for T.

(OR) **(5)**

(ii)Let T be a linear operator on an n-dimensional vector space V. Let A b an $n \times n$ matrix. Then prove that characteristic and minimal polynomials for T have the same roots, except for multiplicities.

b. (i) Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1,\dots,c_k$ are the distinct characteristic values of T, then prove that there exist linear operators $E_1,\dots,E_k$ on V such that

(i). $T = c_1E_1 + \dots + c_kE_k$.

(ii). $I = E_1 + \dots + E_k$.

(iii). $E_iE_j = 0, i \neq j$.

(iv). Each E_i is a projection

(v). The range of E_i is the characteristic space for T associated with c_i.

(OR) **(15)**

(ii) State and prove Primary Decomposition theorem.

III. a. (i) If U is a linear operator on a finite dimensional space W, then prove that U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of the minimal polynomial for U.

(OR) **(5)**

(ii) Let T be a linear operator on a vector space V and W a proper T-admissible subspace of V. Prove that W and cyclic subspace $Z(\alpha;T)$ are independent.

b. (i) Let T be a linear operator on a finite-dimensional vector space V. Let p and f be the minimal and characteristic polynomials for T, respectively. Then prove that

(i) p divides f.

(ii) p and f have the same prime factors, except for multiplicities.

(iii) *If* $p = f_1^{r_1} \ldots f_k^{r_k}$ is the prime factorization of p, then $f = f_1^{d_1} \ldots f_k^{d_k}$. where d_i is the nullity of $f_i(T)^{ri}$ divided by the degree of f_i.

(OR) **(15)**

(ii) Let T be a linear operator on a finite-dimensional vector space V and let W_0 be a proper T-admissible subspace of V. There show that exist non-zero vectors $\alpha_1, \ldots, \alpha_r$ in V with respective T-annihilators $p_1, \ldots, p_r$ are such that

(i) $V = W_0 \oplus Z(\alpha_1; T) \oplus \ldots \oplus Z(\alpha_r; T)$;

(ii) p_k *divides* $p_k - 1, k = 2, \ldots, r$.

IV. a. (i) Define the quadratic form q associated with a symmetric bilinear form f and prove that $f(\alpha, \beta) = \frac{1}{4} q(\alpha + \beta) - \frac{1}{4} q(\alpha - \beta)$.

(OR) **(5)**

(ii) Let V be a complex vector space and f be a form on V such that $f(\alpha, \alpha)$ is real for every α. Then f is Hermitian.

b. (i) Let V be a finite-dimensional inner product space and f a bilinear form on V. Then prove that there is a unique linear operator T on V such that $f(\alpha, \beta) = (T\alpha|\beta)$ for all α, β in V, and the map $f \rightarrow T$ is an isomorphism of the space of forms onto $L(V, V)$. **(8)**

(ii) For any linear operator T on a finite-dimensional inner product space V, prove that there exists a unique linear operator T^* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all α, β in V. **(7)** **(OR)**

(iii)Let V be an inner product space and let $\beta_1, \beta_2, \ldots, \beta_n$ be any independent vectors in V. Then construct orthogonal vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in V such that for each $k = 1, 2, \ldots, n$, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a basis for the subspace spanned by$\{\beta_1, \beta_2 \ldots \beta_k\}$.**(15)**

V. a. (i) Let f be a nondegenerate bilinear form on a finite dimensional vector space V. Prove that the set of all linear operator on V which preserves is a group under the operation composition

(OR) **(5)**

(ii) Define: Bilinear forms, Bilinear function, Quadratic form, Skew Symmetric Bilinear form, Positive forms.

b. (i) If f is a non-zero skew-symmetric bilinear form on a finite dimensional vector space V then prove that there exist a finite sequence of pairs of vectors, $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots (\alpha_k, \beta_k)$ with the following properties:

(i) $f(\alpha_j, \beta_j) = 1, j = 1, 2, \ldots, k.$

(ii) $f(\alpha_i, \alpha_j) = f(\beta_i, \beta_j) = f(\alpha_i, \beta_j) = 0, i \neq j.$

(iii) If W_j is the two dimensional subspace spanned by α_j and β_j, then V=$W_1 \oplus W_2 \oplus \ldots W_k \oplus W_0$ where W_0 is orthogonal to all α_j and β_j and the restriction of f to W_0 is the zero form.

(OR) **(15)**

(ii) Let V be a finite dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $B = \{\beta_1, \beta_2, \ldots \beta_n\}$ for V such that the matrix of f in the ordered basis B is diagonal and

$$f(\beta_i, \beta_j) = \begin{cases} 1, & j = 1, 2, ..r \\ 0, & j > r \end{cases}.$$

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